# Options on Dividends

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First draft: January 2010 - This draft: October 2010

**Abstract**

Motivated by recently increased interest in trading derivatives on dividends, we present a simple, yet efficient equity stock price model with discrete stochastic proportional dividends. The model has a closed form for European option pricing and can therefore be calibrated efficiently to vanilla options on the equity. It can also be simulated efficiently with Monte-Carlo and has fast analytics to aid the pricing of derivatives on dividends. While its efficiency makes the model very appealing, it has the drawback that dividends in this model can become negative.

We present the model and also discuss extensions to stochastic interest rates and jumps.

## 1 Introduction

The recent years have seen an increased interest in trading more directly one of the most basic features of equity stock prices, its dividends. Dividends are comparable to coupons of a bond since they provide the equity holder with a stream of income. The main difference, of course, is that a company can decide how much dividend it will pay, hence the dividend amounts are a function of the company's performance.

In the derivatives worlds, dividends have traditionally been traded either directly via forms of over-the-counter (OTC) equity swaps or implicitly by trading calendar spreads of forwards. The latter in particular allows to trade the realized dividends against the assumed, or "implied", dividends. Starting in 2001, dividend swaps have become a way popular for dealers to hedge their dividend exposure [11]. A dividend swap pays directly the aggregated value of all (gross) dividends a company pays in exchange for the initially agreed strike price.

The next step was to introduce dividend futures on an exchange. The first such product, the "Dow Jones EURO STOXX 50 Index Dividend Futures" was introduced by Eurex in June 2008 [1]. Figure 1 shows the growth of the volume of dividend futures over the last two years.

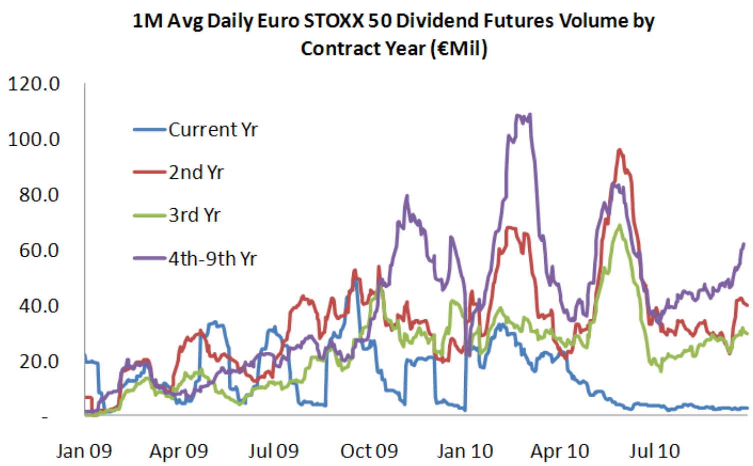


Figure 1: Growth of dividend future volumes since beginning of 2009.

The cumulation of this process so far was the introduction on May 25, 2010 of options on the "S&P500 Annual Dividend Index (DIVD)" and the "S&P 500 Dividend Index (DVS)" on the CBOE [12], and the introduction on the same day of options on both "Euro STOXX50 Index Dividend Futures (FEXD)" and the "Euro STOXX50 Index Dividend Points (DVP)" on Eurex [2]. The exchange traded contracts have the potential to improve price discovery for the volatility market on dividends substantially.

Aside from the listed market, there also the market of OTC derivatives. Here, the potential for structuring ideas around dividends are wide: options on dividends are one way to expresss a view on dividends, but other products such as *Dividend Yield Swaps* and *Knock-Out Dividend Swaps* can give a much more tailored exposure to the realized dividends of an index or a given equity.

At the time of writing, a particularly intriguing observation is the "yield gap" between the yield derived from bonds and the dividend yield on equity. Figure 2 shows that recently, the yield on equity is much better than the yield which can be derived from investing in bonds.

Such a situation can be exploited by entering into rates-dividend hybrid deals, for example into a swap between a rates cash stream vs. a dividend return. We will discuss a few structures in this direction below. This shows the importance of consider dividend investments in the light of rates market.

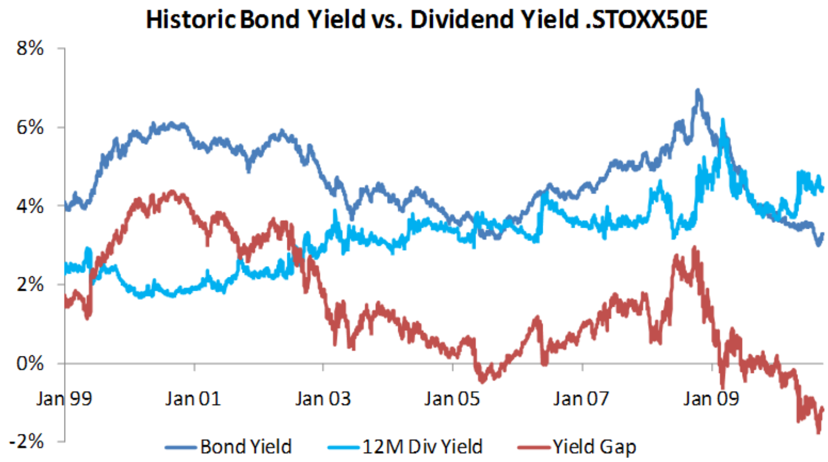


Figure 2: Dividend yield is high in both absolute and relative terms compared to bond yields.

**Pricing with Dividends**

In the light of these developments, it is surprising to find that there is actually not much literature on the modeling of dividends for derivatives pricing and risk management except on the questions on whether these should be proportional to the stock price, fixed in cash or a mixture of the two (these cases have been discussed in depth Bermudez et al. [3] and explained further in Buehler [6]). Beyond this, the only reference to our knowledge on modeling the dividends of an equity is implicitly given by Gaspar's theoretical work [7] on modeling the entire term-structure of the forward curve as a diffusion in an Hilbert-space. This "HJM-approach" is related to a version of our model where a dividend yield is modeled instead of a stream of discrete dividends.

Wit this article, we aim to contribute to the modeling discussion on dividends with a model where dividends are discrete and proportional; the (log-) proportionality factor itself is given by a function of an Ornstein-Uhlenbeck process, meaning that the resulting dividend yield is mean-reverting around a long-term level. While the drawback of using a normal process for the proportionality factor is that it can yield negative dividends, the upside is that the model remains very tractable: pricing vanilla options on the equity for the purpose of calibration, simulating the model with Monte-Carlo and evaluating future forwards are all efficient operations. This gives the model the flavor of a robust workhorse for pricing various variants for options on dividends.

This article is structured as follows: first, we will introduce the setup and the main model. We will also discuss a yield model as a related model. In a second section we discuss main model properties, explain calibration to the equity vanilla option market and show how the model prices options on dividends. We also discuss extending the model to a hybrid stochastic-rates/dividend model, how to incorporate crash jumps and also its relation to Gaspar's framework mentioned above.

An appendix contains most of the actual calculations.

**Acknowledgements**

We are very grateful for the help of Christopher Jordinson, currently at UBS, many of whose ideas are incorporated in the model we present here. His input was crucial to the development of the model.

# 2 Modeling Dividends

We aim to model an equity stock price process $S = (S_t)_t$ under an interest rate process $r = (r_t)_t$ and with effective borrowing costs of $\mu = (\mu_t)_t$. Both are assumed to be deterministic. For notional convenience we write $R_t := e^{\int_0^t r_s - \mu_s \, ds}$ for the drift factor of the equity and $B_t := e^{\int_0^t r_s \, ds}$ for the cash account.[1]

To model the dividend stream of the equity, we assume that the stock is going to pay $N$ random dividends $\Delta = (\Delta^i)_{i=1,\dots,N}$ at dividend dates $0 < \tau_1 < \cdots < \tau_N$.[2] (We will assume that the dividend dates are fixed and known in advance.) Modeling the dividends of the equity as stochastic means modeling each paid dividend $\Delta^i$ as a random variable which is "known" by the dividend time $\tau_i$.[3]

**Dividend Derivatives**

A *dividend swap* on the equity between the dates $T_1$ and $T_2$ pays the accumulated dividends between these two dates against a fixed strike $K$, i.e.

$$\sum_{i:T_1 \le \tau_i \le T_2} \Delta^i - K \ .$$

Similarly, an *option on realized dividends* is a vanilla payoff on the same quantity, i.e. in the case of a call on realized dividends

$$\left( \sum_{i:T_1 \le \tau_i \le T_2} \Delta^i - K \right)^+$$

where we used the notation $x^+ := \max(0, x)$. Note that dividend futures usually pay the realized dividends of the respective underlying index. Hence, an "option

[1]We do not model the possibility of default here, but an extension to the case of a deterministic hazard rate model is well within our framework, c.f. the discussion in Buehler [6].

[2]For ease of exposure we assume that the payment dates are equal to the ex-div dates. An extension to separate ex-div and payment dates is straight-forward.

[3]An exception here is Japan, where dividends are usually announced after the ex-div date.

on a dividend future" is effectively also an option on realized dividends for the period of the future.

More advanced OTC products on dividends are structures, where the payoff of the dividend is linked to the performance of the equity itself. One example is a *knock-out dividend swap*, i.e. a dividend swap which pays each realized dividend $\Delta^i$ against a previously fixed dividend $K^i$ unless *another* equity $\tilde{S}$ trades below a given barrier. The payoff for this product is

$$\sum_{i:T_1 \le \tau_i \le T_2} 1_{\min_{j:j\le i} \tilde{S}_{\tau_j} < B} \left(\Delta^i - K^i\right) \ .$$

Another example is a *dividend yield swap*, which pays the sum of realized dividends over the monthly average spot of the equity:

$$\frac{\sum_{i:T_1 \le \tau_i \le T_2} \Delta^i}{\sum_{k:T_1 \le t_k \le T_2} \sum S_{t_k}}$$

where $(t_i)_i$ are monthly fixings (other variants scale the realized dividends by the stock price at the end of the period, or divide each dividend by the stock price of the previous trading day).

`TODO: enter here idea bond yield vs. div yield and other products`

## The Equity and its Dividends

To model an equity process with a stream of stochastic dividends, the challenge is to find a distribution for the dividends which still allows pricing and risk management of options on the underlying efficiently such that the model can be calibrated easily to observed prices of equity vanilla options.

To approach this topic, we first assume that we face an idealized liquid market of dividend swaps, which allows us to forward-trade any dividend prior to its dividend date. In particular, we assume that there is at any time $t$ a market for the dividends with dividend dates past $t$. We denote the forward-price at time $t$ for the $i$th dividend accordingly by $\Delta_t^i$. As a consequence of our assumptions, its value is given under any risk-neutral measure as

$$\Delta_t^i := \mathbb{E}_t\left[\, \Delta^i \,\right] \ . \tag{1}$$

At this point, it is important to underline that dividends, being basically unknown future non-negative cashflows, are tradable: the process $(\Delta_t^i)_{t\in[0,\tau_i]}$ is a martingale.

With the setup of the previous section, we can deduct a basic shape of any equity model which is consistent with a given dividend stream: basically, the assumptions that dividends can be traded means that the stock price can not drop below the properly discounted value of future dividends at any point. Following the discussion in [6], standard no-arbitrage replication arguments imply that the stock price process *has* to have the form

$$S_t = R_t \left\{ \tilde{S}_0 X_t + \sum_{i:\tau_i > t} \frac{\Delta_t^i}{R_{\tau_i}} \right\}$$

with $\tilde{S}_0 = S_0 - \sum_i \Delta^i / R_{\tau_i}$ for a (local) martingale $X$. Accordingly, the forward at time zero of the process is given as

$$F_t = R_t \left\{ S_0 - \sum_{i:\tau_i \leq t} \frac{\Delta_0^i}{R_{\tau_i}} \right\} .$$

#### Black & Scholes

The standard extension to Black & Scholes' model is the classic case where the dividend is proportional to the equity, i.e.[4]

$$\Delta^i = S_{\tau_i -}(1 - e^{-D_i}) \quad \text{with} \quad D^i := -\ln\left(1 - \frac{\Delta_0^i}{F_{\tau_i -}}\right) . \tag{2}$$

(Note that we also have $\Delta^i = S_{\tau_i}(e^{D_i} - 1)$.) The corresponding stochastic model is then given by

$$S_t = F_t X_t$$

where $X$ is a log-normal model with volatility term structure $\sigma$. Its SDE is given as

$$\frac{dS_t}{S_t} = (r_t - \mu_t)\; dt + \sigma_t \, dW_t - \sum_{i:t_i \leq t} D_i \, \delta_{\tau_i}(dt) \tag{3}$$

(we use $\delta_x(\cdot)$ to denote the Dirac-measure in $x$). In this model, the dividend stream is obviously stochastic – it exhibits the same volatility as the underlying equity.

#### Stochastic Proportional Dividends

The model we want to propose here is given as well in the form

$$\frac{dS_t}{S_t} = (r_t - \mu_t)\, dt + \sigma_t \, dW_t - \sum_i d_i \, \delta_{\tau_i}(dt) \tag{4}$$

but the dividend ratios $d_i$ are random: to model them, we use an Ornstein-Uhlenbeck process

$$dy_t = -\kappa y_t \, dt + \nu \, dB_t$$

and set

$$d_i := (D_i + E_i y_{\tau_i}) + C_i \tag{5}$$

[4]The value of the proportionality factor $\hat{d}^i$ here is derived by using that both $F_{\tau_i -} e^{-D_i} = F_{\tau_i}$ and $F_{\tau_i -} - \Delta^i = F_{\tau_i}$ have to hold.

where $D_i$ is the Black & Scholes value for the proportional dividend as defined in (2). The constant $E_i$ allows to blend between normal and log-normal volatility for the dividend yield by using $E_i = 1$ or $E_i = D_i$, respectively. Finally, the constant $C_i$ is determined by matching the forward such that $\mathbb{E}[S_t] = F_t$ for all $t$; see appendix section A.1 for details.

To motivate our model choice, let us define the "forward yield" between $T_1$ and $T_2$, seen at a time $t$, as the sum of the expected dividends between the two reference times, divided by the current spot level:

$$\bar{y}_t(T_1, T_2) := \frac{\sum_{i:T_1 < \tau_i \leq T_2} \Delta_t^i}{S_t} \ .$$

It is easy to see that in the current model, each dividend has the property that $\log \Delta_t^i / S_t$ is an affine function of $u_t$, which implies that the log-yield $\log \bar{y}_t(T_1, T_2)$ is also approximately proportional to $u_t$.

Looking then at historical data, we indeed see that the log-forward yield seems to mean-revert around some fixed mean. The stable pattern which could be observed during "the great moderation" of the early 2000's has of course been severely disturbed with the onset of the financial crisis and, in particular, Lehman's default in October 2008. The market has since recovered, but to a lower level of implied yield. Figure 3 illustrates this point.

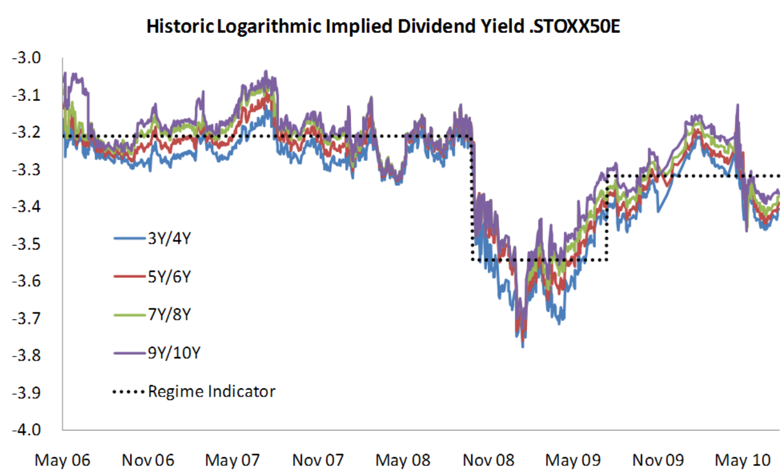


Figure 3: The graph shows historical log-dividend yields for future periods. For example, the 9Y/10Y point refers to the floating maturity forward dividend swap between 9Y and 10Y, divided by the spot at the observation point.

**Model Properties**

The first important observation is that the model's stock price is log-normal: its explicit form is

$$S_t = R_t S_0 e^{\int_0^t \sigma_s\, dW_s - \frac{1}{2}\int_0^t \sigma_s^2\, ds - \sum_{i:\tau_i \le t}\left\{C_i + (D_i+E_i)\left[y_0 e^{-\kappa\tau_i} + \nu \int_0^{\tau_i} e^{-\kappa(\tau_i - s)}\, dB_s\right]\right\}} ,$$

with log-variance

$$\mathrm{Var}(t) = \int_0^t \sigma_s^2\, ds + \sum_{i:\tau_i \le t} (D_i+E_i) \left\{ \sum_{j:j\le i} (D_j + E_j)\nu^2 \frac{1-e^{-2\kappa\tau_j}}{2\kappa} - \rho\nu \int_0^{\tau_i} \sigma_s e^{-\kappa(\tau_i - s)}\, ds \right\} .$$

Since the stock price is log-normal, we can price options on the equity very efficiently using Black & Scholes' formula. This in turn means that the following calibration procedure for the model can be used:

1. First, we determine reasonable values for the dividend volatility term structure via the two parameters $\nu$ and $\kappa$. One possibility of estimating these is by implying them from observed option on dividend prices. A second option is to infer them from historic data and estimate he volatility of implied dividends.

   We also need to determine a reasonable correlation term $\rho$. The correlation term governs the co-movement of the spot and the log-yields.

2. Given the dividend parameters, calibrate the remaining equity volatility $\sigma$ to a term-structure of implied volatilities on the equity: assume that we are given a term structure of reference maturities $0 < T_1 < T_2 < \cdots$ for which we are given Black & Scholes-implied volatilities $(\Sigma_k)_k$. We can then fit a piecewise linear "equity" volatility term structure $\sigma \equiv (\sigma_k)_K$ which is constant over the intervals $[T_{k-1}, T_k]$ by bootstrapping – we start with $\sigma_1$ on $[0, T_1]$ which we determine by solving a quadratic equation using

   $$\mathrm{Var}(T_1) \stackrel{!}{=} \Sigma_1^2 T_1 \ .$$

   Then we iteratively solve for the next value of $\sigma_k$ on the interval $[T_{k-1}, T_k]$ using

   $$\mathrm{Var}(T_k) - \mathrm{Var}(T_{k-1}) \stackrel{!}{=} \Sigma_k^2 T_k - \Sigma_{k-1}^2 T_{k-1}$$

   This shows that the model can be efficiently calibrated to a term structure of vanilla options.

REMARK 2.1 *In actual applications, the above fitting to the implied volatility term structure of the equity along a given strike is usually done "on-the-fly", i.e. within the actual pricer. This means that the model's internal equity volatility $\sigma$ is not actually visible to the user, but they see the model as using the implied volatilities of the equity as a parameter. Consequently, the model is able to produce transparent vega risk with respect to equity implied volatilities.*

Another advantage of the model which derives from its normal form is that it is possible to efficiently calculate the forward price of the equity as a function of spot and the driving Ornstein-Uhlenbeck process $u$ at a given future time,

$$F_t(T) := \mathbb{E}_t[\, S_T \,]$$

(see appendix A.2 below). This is important because it allows us[5] computing future expected dividend values using

$$\Delta_t^\ell = F_t(\tau_{\ell-}) - F_t(\tau_\ell) = F_t(t \vee \tau_{\ell-1}) \frac{R_{\tau_\ell}}{R_{t\vee\tau_\ell - 1}} - F_t(\tau_\ell)$$

(we used the notation $x \vee y := \max(x, y)$). In other words, the model allows efficient access to pricing dividend swaps analytically within a Monte-Carlo simulation. Hence, we can use the model to price and risk manage not only options on realized dividends, but also on more involved trades which depend in value on futures on dividends.

### Negative Dividends

While these properties are clearly very appealing, the model's structure also reveals one of its main drawbacks: the fact that the dividends themselves can become negative. This happens if $d_i$ becomes negative which, in the light of (5), is a possible event.

The probability of this happens at a dividend date $\tau_i$ is

$$\mathbb{P}\left[ Y \frac{1 - e^{-2\kappa\tau_i}}{2\kappa} \le \frac{D_i + C_i}{E_i} - u_0 e^{-\kappa\tau_i} \right]$$

where $Y$ is standard normal.

However, we think that the advantage of analytical tractability of the model far outweighs the downside of having potentially negative dividends.

## 2.1 Using the Model

For most derivatives on dividends, we will need to revert to numerical methods to compute the value of the payoff and risk manage it. We will here discuss briefly how to implement an efficient Monte-Carlo and will show how the model prices some options on dividends.

### Monte-Carlo Simulation

In order to simulate the model (4), we assume that we have some "observation dates" $0 < t_1 < \cdots < t_n$ of interest at which points we wish to evaluate our payoff; for ease of exposure we assume w.l.g. that these dates include all dividend dates in the respective period. One convenient feature of our essentially normal

[5] We use the fact that by definition $F_t(T) = R_T/R_t(S_t - \sum_{i:t\le\tau_i\le T} R_t/R_{\tau_i}\Delta_t(i))$.

model is that we do not need to use Euler to simulate our SDE because the model is in fact normal between two of the observation dates.

To see this, note that

$$\begin{cases} y_{t_{k+1}-} & = & y_{t_k} e^{-\kappa(t_{k+1}-t_k)} + \int_{t_k}^{t_{k+1}} e^{-\kappa(t_{k+1}-s)} \theta_s \, ds + \nu \int_{t_k}^{t_{k+1}} e^{-\kappa(t_{k+1}-s)} \, dB_s \\ \log S_{t_{k+1}-} & = & \log S_{t_k} + \int_{t_k}^{t_{k+1}} \sigma_s \, dW_s + \int_{t_k}^{t_{k+1}} \left( r_s - \mu_s - \frac{1}{2}\sigma_s^2 \right) ds \\ & & -(D_{k+1} + E_{k+1} y_{\tau_{k+1}} + C_{k+1}) \end{cases}$$

(We used a slightly lax notation here: the $D_{k+1}$, $E_{k+1}$ and $C_{k+1}$ are either the relevant coefficients from a dividend payable at $\tau_{k+1}$ or zero.)

In other words, the joint proces $(S, y)$ is normal on the interval $[t_k, t_{k+1}]$ and we can take one large step using only two random variables. Using now the previously mentioned ability to compute dividend swaps on the fly in the Monte-Carlo, we can display the evolution of the dividend swap fair strikes over time on a Monte-Carlo path.

#### Pricing Options on Dividends

`Some example prices coming here`

## 2.2 Extensions and Related Models

We briefly want to give some overview over possible extensions to the current model.

#### Stochastic Interest Rates

An interesting aspect of dividend modeling has to be the relation ship between dividends and interest rates. From an economic point of view, both rates products and dividend-paying equity play a similar role in providing investors with a stream of income. Consequently, we can assume that the market's dividend yield is related to the level of interest an investor can earn by buying just a coupon-bearing bond.

While we will not discuss the economic situation much further here, we would like to point out that the current model can easily be adapted to the standard Hull & white interest model [9], and that it retains its analytical tractability since the distribution of the stock price remains log-normal.

To start with, let us assume that we can observe at time zero a market of zero coupon bonds $(P_t)_t$. We are going to model the interest rate using an Ornstein-Uhlenbeck process

$$dr_t = (\theta_t - q r_t) \, dt + \xi_t \, dW_t^r$$

While mean-reversion speed $q$, the interest volatility $\xi$ are free parameters which have to be calibrated from the swaption market, the level of mean-reversion, $\theta$, is calibrated to the observed zero coupon bond term structure by stipulating

$$\mathbb{E}\left[ e^{-\int_0^t r_s \, ds} \right] \stackrel{!}{=} P_t$$

(in practical applications, one would never actually calculate $\theta$ but use it only in integrated form; see [3]).

Given our Hull & white model, we can now define the equity process via, once again,

$$\frac{dS_t}{S_t} = (r_t - \mu_t)\,dt + \sigma_t\,dW_t - \sum_i d_i\,\delta_{\tau_i}(dt)$$

with $d_i$ given again as in (5). Since $r$ and therefore any integral over it are normal, it follows that the stock price process in this joint Hybrid model is still log-normal. Consequently, the analytic tractability of the model is preserved and it is a very convenient candidate if we want to price joint rates-dividend derivatives. We even maintain the ability to run an efficient Monte-Carlo by using the methods discussed in Brockhaus et.al. [5], page 36.

Remark 2.2 *Both our original and our rates/dividends hybrid model allow the introduction of several Ornstein-Uhlenbeck factors to drive the respective curves; see also Hull's description for the 2-factor rates model in [8].*

### Jump Risk, Crash Risk, Credit Risk

Handling plain credit risk within a deterministic intensity model is straight forward following the recipe in [6]. If one is interested in adding less drastic jumps to the equity process which also impact the dividend, then the classis approach by Merton [10]. One point of consideration is the behavior of the yield if the equity exhibits a sudden drop. We would assume that a *sudden* drop in the spot price will also lead to a drop in yield. This can be seen very well in figure 3 on page 7 which shows that the stock market drop in October 2008 lead to a substantial drop in dividend yield as well.

To model such a relationship, let us consider series of normal crash or jump scenerios $(\gamma_j)_{j=1,\dots,M}$ for the stock price process, each with mean $m_j$ and volatility $s_j$, and driving Poisson processes $(N^j)_j$ with crash intensities $(\lambda_j)_j$. We also assume we have corresponding shock scernarios $(g_j)_j$ for the dividend yield.

The diffusion for the stock price is then

$$\frac{dS_t}{S_t} = (r_t - \mu_t)\,dt + \sigma_t\,dW_t - \sum_i d_i\,\delta_{\tau_i}(dt) - \left(\sum_{j=1}^{M} \gamma_j dN_t^j + \tilde{\lambda}_j\,dt\right) .$$

with $\tilde{\lambda}_j := \lambda_j(e^{-m_j+\frac{1}{2}s_j^2} - 1)$. In order to incorporate synchronous jumps into the dividend yield as well, we additionally rewrite our driving Ornstein-Uhlenbeck process as

$$dy_t = -\kappa y_t\,dt + \nu\,dB_t - \sum_{j=1}^{M} g_j dN_t^j .$$

This model has the desired properties of synchronous crashes in equity and yields while it maintains enough flexibility to be adapted to "small" jumps which do not affect the yield itself (they will affect the dividends being proportional to the spot price level). Note that one drawback of incorporating negative jumps into $u$ is that the proportional dividend factors $d_i$ are more likely to become negative.

In terms of analytical tractability this model is once again very convenient: conditional on the number of jumps for each Poisson-process this model has a log-normal distribution which means that it can be used efficiently using Fourier-based pricing for European options.

**Stochastic Yield Dividends**

One of the main features of our "stochastic proportional dividend" model is that it keeps the dividend payments to exactly the payments found in the market data. An alternative is to model a yield on top of the prevalent forward curve: such a model is given by

$$\frac{dS_t}{S_t} = (r_t - \mu_t - y_t - \bar{C}_t)\,dt + \sigma_t\,dW_t - \sum_i D_i\,\delta_{\tau_i}(dt)$$

where the $D_i$ are the proportional dividends from the Black & Scholes formulation (3), and where the function $\bar{C}$ is chosen such that the model fits the forward, i.e. $\mathbb{E}[S_t] = F_t$.[6] While being numerically easier to handle than the proportional dividend model above, it has the disadvantage that the yield will often be negative over periods between the input market dividend dates. As such, the model might be more useful if the equity forward itself is given as a yield – in fact, this model is related to Gaspar's HJM-framework [7] for forward curves which we mentioned in the introduction.

We will not discuss the details of this approach further, but the interested reader will find that most calculations are either similar or simpler compared to our proportional dividend model.

# 3 Conclusions

We have discussed a stochastic dividend model with very tractable analytics for calibration towards the vanilla market and for the pricing and risk management of dividend derivatives. To the best of our knowledge, this is the first attempt to model the dividend stream of an equity in a consistent manner for the purpose of derivatives pricing. The model's simplicity qualifies it as a standard tool in a derivatives library, while the fact that its dividend are not always positive means that there is clearly a further need for development in the dividend modeling area.

[6] I.e. we use the relation $e^{\int_0^t \bar{C}_s\,ds} = \mathbb{E}\left[e^{\int_0^t \sigma_s\,dW_s - \frac{1}{2}\int_0^t \sigma_s^2\,ds - \int_0^t y_s\,ds}\right]$ to find $\bar{C}$.

## A.1 Matching the Forward

Fix some $\ell$. We want to compute

$$(*) = \mathbb{E}_t\left[ e^{\int_0^{\tau_\ell}\sigma_t\, dW_t - \frac{1}{2}\int_0^{\tau_\ell}\sigma_t^2\, dt - \sum_{j:j\leq\ell} d_j} \right] \ .$$

As a first step, we change into the "equity measure" $\mathbb{Q}$ under which

$$dy_t = (\theta_t - \kappa y_t)\ dt + \nu\, dB_t \ .$$

with $\theta_t := \sigma_t \rho \nu$. under this measure, we have

$$(*) = \mathbb{E}^{\mathbb{Q}}\left[ e^{-\sum_{j:j\leq\ell} d_j} \right] \ .$$

Since we have $de^{\kappa t} y_t = e^{\kappa t}\theta_t\, dt + e^{\kappa t}\nu\, dB_t$, we get

$$y_t = y_u e^{-\kappa(t-u)} + \int_u^t e^{-\kappa(t-s)}\theta_s\, ds + \nu\int_u^t e^{-\kappa(t-s)}\, dB_s \ . \tag{6}$$

That means that for any $p$

$$\mathbb{E}^{\mathbb{Q}}\left[ e^{-py_t} \,\middle|\, \mathcal{F}_u \right] = e^{-py_u K(t,u) - p\Theta(t,u) + p^2\Gamma(t,u)} \ . \tag{7}$$

with

$$\begin{cases} \Theta(t,u) & := & \int_u^t e^{-\kappa(t-s)}\theta_s ds \\ K(t,u) & := & e^{-\kappa(t-u)} \\ \Gamma(t,u) & := & \frac{1}{2}\nu^2 \dfrac{1-e^{-2\kappa(t-u)}}{2\kappa} \ . \end{cases} \tag{8}$$

We also abbreviate

$$\Theta_j := \Theta(\tau_j, \tau_{j-1}) \quad , \quad K_j := K(\tau_j, \tau_{j-1}) \quad \text{and} \quad \Gamma_j := \Gamma(\tau_j, \tau_{j-1}) \ .$$

**Iteration**

The aim is to make sure with out choice of $(C_j)_{j=1,\ldots,N}$ that

$$1 \stackrel{!}{=} \frac{\mathbb{E}^{\mathbb{Q}}\left[ e^{-\sum_{j:j\leq\ell} d_j} \right]}{e^{-\sum_{j:j\leq\ell} D_j}} = \mathbb{E}^{\mathbb{Q}}\left[ e^{-\sum_{j:j\leq\ell}(E_j y_{\tau_j} + C_j)} \right] \ .$$

Consider the formula

$$c_\ell(p) := \log \mathbb{E}^{\mathbb{Q}}\left[ e^{-\sum_{j:j<\ell}(E_j y_{\tau_j} + C_j) - (E_\ell + p) y_{\tau_\ell}} \right]$$

which obviously yields with $C_\ell := c_\ell(0)$ the desired correction terms. We will derive these functions iteratively: we start with $\ell = 1$. Using (7) yields readily

$$c_1(p) = -(E_1 + p)K_1 y_0 - (E_1 + p)\Theta_1 + (E_1 + p)^2\Gamma_1 \ .$$

and therefore

$$C_1 := c_1(0) \ .$$

The next step is $\ell > 1$ to determine $c_\ell(p)$ assuming we know $c_j(q)$ and therefore $C_j$ for $j < \ell$.

$$\begin{aligned}
e^{c_\ell(p)} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{j:j<\ell}(E_j y_{\tau_j}+C_j)-(E_\ell+p)y_{\tau_\ell}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{j:j<\ell}(E_j y_{\tau_j}+C_j)}\ \mathbb{E}^{\mathbb{Q}}_{\tau_\ell}\left[e^{-(E_\ell+p)y_{\tau_\ell}}\right]\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{j:j<\ell}(E_j y_{\tau_j}+C_j)} e^{-(E_\ell+p)K_\ell y_{\tau_{\ell-1}}-(E_\ell+p)\Theta_\ell+(E_\ell+p)^2\Gamma_\ell}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{j:j<\ell-1}(E_j y_{\tau_j}+C_j)-(E_{\ell-1}+(E_\ell+p)K_\ell)y_{\tau_{\ell-1}}}\right]\ e^{-(E_\ell+p)\Theta_\ell+(E_\ell+p)^2\Gamma_\ell-C_{\ell-1}} \\
&= e^{-(E_\ell+p)\Theta_\ell+(E_\ell+p)^2\Gamma_\ell+c_{\ell-1}((E_\ell+p)K_\ell)-C_{\ell-1}}
\end{aligned}$$

which means that

$$c_\ell(p) = -(E_\ell+p)\Theta_\ell + (E_\ell+p)^2\Gamma_\ell + c_{\ell-1}\left((E_\ell+p)K_\ell\right) - C_{\ell-1} \qquad (9)$$

is well-defined.

However, given the iterative nature of this formula, the growth for the computational overhead is quadratic – in other words, the model discussed here cannot reasonably be used in this form for indices which have a lot of discrete dividends. That said, in such a case a more yield-based model such as Gaspar's might be more suitable.

## A.2 Future Forwards

A very similar calculation as the above allows us to compute the future forwards of the model,

$$F_t(T) := \mathbb{E}_t[\,S_T\,] \ .$$

We sketch the idea here: let $n : \tau_n \le T < \tau_{n+1}$. First of all,

$$F_t(T) = S_t \frac{R_T}{R_t}\ \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\sum_{j:j\le n}(E_j y_{\tau_j}+C_j)}\right] \ .$$

The last term can be handled with the same method as above: let $k : \tau_{k-1} \le t < \tau_k$. Then define for $\ell : \ell \le n$

$$c_\ell(t;p) := \mathbb{E}^{\mathbb{Q}}_t\left[e^{-\sum_{j:k\le j<\ell}(E_j y_{\tau_j}+C_j)-\left((E_\ell+p)y_{\tau_\ell}+C_\ell\right)}\right] \ .$$

The first term we need to know is

$$c_k(t;p) := -(E_k+p)K(t,\tau_k)y_t - (E_k+p)\Theta(t,\tau_k) + (E_k+p)^2\Gamma(t,\tau_k) - C_k \ .$$

and all further terms have the same structure as (9), i.e.

$$c_\ell(t;p) = -(E_\ell+p)\Theta_\ell + (E_\ell+p)^2\Gamma_\ell + c_{\ell-1}\left(t;(E_\ell+p)K_\ell\right) - C_{\ell-1} \ .$$

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